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STEPANOV ALMOST PERIODIC SOLUTIONS OF CLIFFORD-VALUED NEURAL NETWORKS

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Abstract. We introduce Clifford-valued neural networks with leakage delays. Furthermore, we study the uniqueness and existence of Clifford-valued Hopfield artificial neural networks having the Stepanov weighted pseudo almost periodic forcing terms on leakage delay terms. However the noncommutativity of the Clifford numbers' multiplication made our investigation diffcult, so our results are obtained by decomposing Clifford-valued neural networks into real-valued neural networks. Our analysis is based on the differential inequality techniques and the Banach contraction mapping principle.

1. Introduction

In the past decades, the dynamics of various neural networks have been extensively studied. Many kinds of neural networks such as Hopfield neural networks and cellular neural networks etc., have received much more attention from many fields ([9],[12],[14],[16]). They are a good tool for the approximation of dynamical systems, and so their successful application requires an understanding of their long term behavior with dynamical properties, in specially, their existence, uniqueness and stability.

The mathematical theory that enables machine learning of artificial intelligence is Kolmogorov-Arnold theorem[6], which is the starting point of neural network models. A sufficiently large function space can be constructed by choosing a suitable activation function and repeating only this function and arithmetic operation.

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It is known that, as a generalization of real-valued neural networks, the research of complex-valued and quaternion-valued neural networks have been investigated in several kinds of neural networks have attracted more and more attention due to have more advantages than real-valued neural networks in many aspects. However they are sometimes inapplicable for some for some engineering problems for instance such as neural computing, computer and robot vision, image and signal processing. For this reason, researchers attempted recently a more general and complicated neural networks, which is Clifford-valued neural networks. Clifford-valued neural networks are a kind of neural networks whose state variables, connection weights and external inputs are Clifford numbers. They are generalizations of real-valued, complex-valued and quaternion-valued neural networks. However, because the multiplication of quaternion numbers does not satisfy the commutative law. In order to avoid the non-commutativity of the quaternion multiplication, researchers decomposed given system into real-valued systems.

In recent years, C. Xu and P. Li [12] investigate the pseudo almost periodic solution of the following Hopfield neural networks with time-varying leakage delays:

$$\begin{aligned} x'(t) &= -c_i(t)x_i(t-\eta_i(t)) + \sum_{j=1}^{n} a_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) \qquad (1) \\ &+ \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)g_j(x_j(t-\sigma_{ijl}(t)))g_l(x_l(t-\mu_{ijl}(t))) + I_i(t). \end{aligned}$$

The initial conditions associated with system (1) are of the form

$$x_i(s) = \varphi_i(s),\ s \in [-\tau, 0],\ \varphi_i \in C([-\tau, 0], \mathcal{A}),\ i \in I,$$

Motivated by the aforementioned works, to illustrate our abstract result, we investigate and establish some sufficient conditions to guarantee the existence and uniqueness of Stepanov-like weighted pseudo almost periodic solutions of Hopfield neural network for the system (1) on Clifford algebra as follow: n is the number of units in a neural network, $x_i(t) \in \mathcal{A}$, which is known as Clifford number, corresponds to the state vector of the i-th unit at time t, $c_i(t) > 0$ represents the rate with which the i-th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs, $a_{ij}(t), b_{ijl}(t) \in \mathcal{A}$ are first-order and second-order connection weights of the natural network, $\eta_i(t) > 0$ and $\tau_{ij}(t), \sigma_{ijl}(t), \nu_{ijl}(t) > 0$ correspond to the leakage and transmission delays, respectively, $I_i(t) \in \mathcal{A}$ denotes the external inputs at time t, and $f_j, g_j : \mathcal{A} \to \mathcal{A}$ is the activation function for signal transmission of the i-th neuron.

2. Preliminaries and notations

The Clifford algebra was establishment by the British mathematician William K. Clifford in 1878 which is a generalization of the plural, quaternion, and Glassman algebra.

First, we introduce the definition and properties of Clifford algebra which is well known. We shall refer to [8],[14],[12] and references therein.

Clifford algebra over $\mathbb{R}^n$ is defined as $\mathcal{A} = \{\sum_{A\subset\{1,2,3,\cdots,m\}} a^A e_A, a^A \in \mathbb{R}\}$ where $e_A = e_{h_1}e_{h_2}\cdots e_{h_\mu}$, with $A = \{h_1, h_2, \cdots, h_\mu\}$, $1 \le h_1 < h_2 < h_2 < \cdots < h_\mu \le m$.

Moreover, $e_\emptyset = e_0 = 1$ and $e_{\{h\}} = e_h$, $h = 1, 2, \cdots, m$ are called Clifford generators which satisfy the Hamilton's multiplication rules ; the relations $e_i^2 = -1$ and $e_ie_j + e_je_i = 0$, $i = j$, $i, j = 1, 2, 3, \cdots, m$. For simplicity, when one element is the product of multiple Clifford generators, we will write its subscripts together. For example, $e_1e_2 = e_{12}$ and $e_3e_7e_2e_5 = e_{3725}$. We define $\Delta = \{\emptyset, 1, 2, \cdots A, \cdots, 12\cdots m\}$ then it is easy to see that $\mathcal{A} = \{\sum_A a^A e_A, a^A \in \mathbb{R}\}$, where $\sum_A$ is a brief form of $\sum_{A\in\triangle}$ and $dim_{\mathbb{R}}\mathcal{A} = \sum_{k=o}^{m}\binom{m}{0} = 2^m$.
For any Clifford number $x = \sum_A x^A e_A \in \mathcal{A}$, the involution of x is defined as $\bar{x} = \sum_A a^A \bar{e}_A$ where $\bar{e}_A = (-1)^{\frac{\sigma[A](\sigma[A]+1)}{2}}$ and

$$\sigma[A] = \begin{cases} 0, \text{ if } A = \emptyset \\ \mu, \text{ if } A = h_1h_2\cdots h_\mu. \end{cases}$$

From the definition, it is directly deduced that $e_A\bar{e}_A = \bar{e}_Ae_A = 1$. Moreover, For Clifford-valued function $x = \sum_A x^A e_A$ where $x^A : \mathbb{R} \to \mathbb{R}$, $A \in \mathcal{A}$, and its derivative is given by $\frac{dx(t)}{dt} = \sum_A \frac{dx^A}{dt} dte_A$. Since $e_B\bar{e}_A = (-1)^{\frac{\sigma[A](\sigma[A]+1)}{2}} e_Be_A$, we can write $e_B\bar{e}_A = e_c$ or $e_B\bar{e}_A = -e_c$, where e_c is a basis of Clifford algebra $\mathcal{A}$. For example, $e_{h_1h_2}\bar{e}_{h_2h_3} = -e_{h_1h_2}e_{h_2h_3} = -e_{h_1}e_{h_2}e_{h_3} = -e_{h_1}(-1)e_{h_3}e_{h_1}e_{h_3} = e_{h_1h_3}$. Hence it is possible to find a unique corresponding basis e_c for the given $e_B\bar{e}_A$.
Define

$$\sigma[B\cdot\bar{A}] = \begin{cases} 0, \text{ if } e_B\bar{e}_A = e_c \\ \mu, \text{ if } e_B\bar{e}_A = -e_c \end{cases}$$

and then $e_B\bar{e}_A = (-1)^{\sigma[B\cdot\bar{A}]}e_c$.

In addition, for any $g \in \mathcal{A}$, we can find g^c a unique satisfying $g^{B\cdot\bar{A}} = (-1)^{\sigma[B\cdot\bar{A}]}g^c$ for $e_B\bar{e}_A = (-1)^{\sigma[B\cdot\bar{A}]}e_c$. Hence $g^{B\cdot\bar{A}}e_B\bar{e}_A = g^{B\cdot\bar{A}}(-1)^{\sigma[B\cdot\bar{A}]}$ $e_C = (-1)^{\sigma[B\cdot\bar{A}]}g^C b(-1)^{\sigma[B\cdot\bar{A}]}e^C = g^C e_C$ and $g = \sum_C g^C e_C \in \mathcal{A}$.

Remark 1.Clifford-valued system (1) includes real-valued systems and complex-valued systems as its special cases. In fact system (1), when m, the number of the generators of A, equals $m = 0$, $m = 1$ and $m = 2$, system (1) degenerates into real-valued, complex-valued, and quternion-valued systems as its special cases, respectively. And so, in many respects, Clifford-valued system model is far more advantages than the system model in [13],[15].

Secondly, let $(X, ||\cdot||)$ be a Banach space and $BC(\mathbb{R}, X)$ be the set of all bounded continuous functions from $\mathbb{R}$ to X. For a given $T > 0$ and each $\rho(weights)$, let $\mu(T,\rho)= \int_{-T}^{T}\rho(t)dt$. Furthermore, we review some definitions and lemmas well known from our references([1],[4],[12],[17]) and references therein.

Definition 2.1. A function $f \in BC(\mathbb{R}, \mathcal{A})$ is called *almost periodic* on $\mathcal{A}$ if for every $\epsilon > 0$, if there exists an $l > 0$ such that every interval of length $l(\epsilon)$ contains a number τ with property that

$$||f(t+\tau) - f(t)|| < \epsilon, \text{ for every } t \in \mathbb{R}.$$

The collection of such functions is denoted by $AP(\mathbb{R}, \mathcal{A})$.

Definition 2.2. The Bochner transform $f^b(t,s),\ t \in \mathbb{R},\ s \in [0,1]$ of a function $f : \mathbb{R} \to X$ is defined by $f^b(t,s) := f(t+s)$.

Definition 2.3. Let $p \in [1,\infty)$. The space $BS^p(\mathbb{R}, \mathcal{A})$ of all Stepanov bounded functions, with the exponent p consists of all measurable functions $f : \mathbb{R} \to \mathcal{A}$ such that $f^b \in L^p(\mathbb{R}; L^p((0,1), \mathcal{A}))$. This is a Banach space with the norm

$$||f||_{S^p} := \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} ||f(\tau)||^p d\tau\Big)^{\frac{1}{p}}.$$

We define the Stepanov weighted ergodic space, for $f \in BC(\mathbb{R}, \mathcal{A})$,

$$\begin{aligned} & PAP_0(L^p([0,1], \mathcal{A}), \rho) \\ = \ & \{f; \lim_{T\to\infty} \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \Big(\int_t^{t+1} ||f(s)||^p ds\Big)^{\frac{1}{p}} \rho(t)dt = 0\}. \end{aligned}$$

Definition 2.4. A function $f \in BS^p(\mathbb{R}, \mathcal{A})$ is said to be a S^P-weighted pseudo-almost periodic if it can be expressed as $f = h + \varphi$, where $h^b \in AP(\mathbb{R}, (L^p([0,1], \mathcal{A}))$, $\varphi^b \in PAP_0(\mathbb{R}, L^p([0,1], \mathcal{A}), \rho)$.

The collection of such functions will be denoted by $PAP(\mathcal{A},\rho,p) = PAP(L^p([0,1],\mathcal{A}),\rho)$ which is a closed subspace of $BC(\mathbb{R},L^p([0,1],\mathcal{A}))$ relatively to the norm $||\cdot||_{S^p}$, and therefore is a Banach space.
Note that $f \in S^pWPP_0(\mathbb{R},\mathcal{A},\rho)$ if and only if $f^b \in WPP_0(\mathbb{R},L^p([0,1],\mathcal{A})$ $,\rho)$.

Theorem 2.5. [2] *For Banach space X,Y, let $f \in C(\mathbb{R}\times X : Y)$ be a almost periodic in $t \in \mathbb{R}$ uniformly in $y \in K$ where $K \subset Y$ ia any compact subset $K \subset Y$. Then the superposition operator N_f defined by:*

$$N_f : AP(\mathbb{R},X) \to AP(\mathbb{R},Y),\ N_f(u) := [t \to (f(t),u(t)]$$

is well defined continuous from $AP(\mathbb{R},X)$ into $AP(\mathbb{R},Y)$.
Furthermore, $u \in AP(\mathbb{R},X)$. Then we have $[t \to f(t,u(t))] \in AP(\mathbb{R},Y)$.

Lemma 2.6. *If $\varphi(\cdot) \in PAPS^p(\mathbb{R},\mathcal{A}), \tau(\cdot) \in APS^p(\mathbb{R},\mathcal{A})$, then $\varphi(\cdot,-\tau(\cdot)) \in PAPS^p(\mathbb{R},\mathcal{A})$.*

Definition 2.7. A function $f = \sum_{i=1}^n f^A e_A : \mathbb{R} \to \mathcal{A}$ is said to be Stepanov almost periodic, if $f^A \in S^pAP(\mathbb{R},\mathbb{R})$ for all $A \in \mathcal{A}$.

Similar to the proof of Lemma 3.7 in [10], one can prove:

Lemma 2.8. *Let $f \in C(\mathbb{R},\mathcal{A})$ and satisfy the Lipschiz condition. If $g \in S^pAP(\mathbb{R},L^p((0,1),\mathcal{A}))$, then $f(g(x)) \in S^pAP(\mathbb{R},L^p((0,1),\mathcal{A}))$.*

Noting that $M[a_i] > 0$, using the theory of exponential dichotomy in [5], we can easily get:

Lemma 2.9. *For $i = 1,2,3,\cdots,n,\ a_i \in BC(\mathbb{R},\mathbb{R})$ with $\inf_{t\in\mathbb{R}} a_i(t) > 0$. If $f \in BC(\mathbb{R},\mathbb{R}^n)$, then the linear system*

$$x^{'}(t) = A(t)x(t) + f(t)$$

has a unique bounded solution

$$x(t) = \int_{-\infty}^{t} e^{\int_s^t A(u)du} f(s)ds,$$

where $A(t) = diag(-a_1(t),-a_2(t),\cdots,-a_n(t))$.

3. Existence results for Stepanov weighted pseudo almost periodic solution

To avoid the difficulty for the non-commutativity of multiplication of Clifford numbers, firstly we transform the Clifford-valued system (1)

into the real-valued system which is easily to handle ([8], [14], [12]).
This can be established using by $e_A\bar{e}_A = \bar{e}_Ae_A = 1$ and $\bar{e}_Ae_A = e_B$.
For any $g \in \mathcal{A}$, we can find g^c a unique satisfying $g^{B\cdot\bar{A}} = (-1)^{\sigma[B\cdot\bar{A}]}g^c$ for $e_B\bar{e}_A = (-1)^{\sigma[B\cdot\bar{A}]}e_c$. So $g^{B\cdot\bar{A}}e_B\bar{e}_A = g^{B\bar{A}}(-1)^{\sigma[B\cdot\bar{A}]}e_C = (-1)^{\sigma[B\cdot\bar{A}]}$ $g^Cb(-1)^{\sigma[B\cdot\bar{A}]}e^C = g^Ce_C$ and $g = \sum_C g^Ce_C \in \mathcal{A}$.
By decomposing (1) into $x = \sum_A x^Ae_A$, we obtain

$$\begin{aligned} x_i'^A(t) &= -c_i(t)x_i^A(t-\eta_i(t)) + \sum_{j=1}^n\sum_B a_{ij}^{A\cdot\bar{B}}(t)g_j^B(x_j(t-\tau_{ij}(t))), \\ &+ \sum_{j=1}^n\sum_{l=1}^n\sum_{B\in\Lambda} b_{ijl}^{A\cdot\bar{B}}(t)g_j^B(x_j(t-x_{ijl}(t)))g_l^B(x_l(t-\mu_{ijl}(t))) \\ &+ I_i^A(t),\ x_i^A(s) = x_i^A(s),\ s\in[-\tau,0],\ i\in I, \qquad (2), \end{aligned}$$

where

$$x_i(t) = \sum_A x_i^A(t)e_A,\ I_i(t) = \sum_A I_i^A(t)e_A,$$

$$\begin{aligned} &a_{ij}(t) = \sum_A a_{ij}^C(t)e_C,\ a_{ij}^{A\cdot\bar{B}}(t) = (-1)^{n[A\cdot\bar{B}]}a_{ij}^C(t), \\ &b_{ijl}(t) = \sum_A b_{ijl}^C(t)e_C,\ b_{ijl}^{A\cdot\bar{B}}(t) = (-1)^{n[A\cdot\bar{B}]}b_{ijl}^C(t), \\ &g_j(x_j(t-\tau_{ijl}(t))) = \sum_{B\in\lambda} g_j^B(\varphi_j^{C_1}(t-\tau_{ijl}(t)), \varphi_j^{C_2}(t-\tau_{ijl}(t)),\cdots, \\ &x_j^{C_{2m}}(t-\tau_{ijl}(t)))e_B. \end{aligned}$$

Remark 2. It is clear that if $x = (x_1^0, x_1^1,\cdots,x_1^{1\cdot 2\cdots m}, x_2^0, x_2^1,\cdots,x_2^{1\cdot2\cdots m}, \cdots, x_n^0, x_n^1,\cdots,x_n^{1\cdot2\cdots m})^T : \{x_i^A\}$ is a solution to system (2), then $x = (x_1, x_2,\cdots x_n)^T$ must be a solution to (2), where $x_i = \sum_A x_i^Ae_A, A\in\Delta$.

For the sake of convenience to work (3) we established some hypothesis and sufficient criteria, which will be used in this paper, as following:

(H_1) For $i,j,l\in I$ and $A,B\in\Delta$, $c_i(t)\in S^pWPAP(\mathbb{R},\mathcal{A})$, $a_{ij}^{A\cdot\bar{B}}(t)$, $b_{ij}^{A\cdot\bar{B}}(t), I_i(t)\in S^pWPAP(\mathbb{R},\mathcal{A}), \tau_{ij}(t),\sigma_{ijl}(t),\mu_{ijl}(t)\in S^pWPAP(\mathbb{R},\mathcal{A})$.

(H_2) For any $u,v\in A$, functions $f_j^B, g_j^B\in C(\mathcal{A},\mathbb{R})$,there exist positive constant L_j^f, L_j^g such that

$$||f_j^B(u) - f_j^B(v)|| \le L_j^f\sum_{C\in A}||u^C - v^C||,\ f_j^B:\mathbb{R}^{2^m}\to\mathbb{R},$$

$$||g_j^B(u) - g_j^B(v)|| \le L_j^g \sum_{C \in A} ||u^C - v^C||, \; f_j^B : \mathbb{R}^{2^m} \to \mathbb{R}.$$

Additionally, we suppose that $f_j^B(0) = g_j^B(0) = 0$.

(H_3) Let $\mathcal{D} = \{\varphi : \varphi \in S^pWPAP(\mathbb{R}, \mathcal{A})\}$, $||x||_{S^p} = \max_{i \in I} \{\max_{A \in \Delta} |x_i^A|_{S^p}\}$ and $\varphi_0 = \{(\varphi_0)_i^A\}$, where $|x_i^A|_{S^p} = \sup_{t \in \mathbb{R}} (\int_t^{t+1} |x_i^A(s)|^p \, ds)^{\frac{1}{p}}$, $(\varphi_0)_i^A(t) = \int_{-\infty}^t e^{-\int_s^t a_i(u)du} I_i^A(s)ds$, respectively. It is clear that $\mathcal{D}$ is a Banach space.

(H_4) For $i \in I$, there is a function $\tilde{a}_i \in BC(\mathbb{R}, (0, +\infty))$ and a constant $K_i > 0$ satisfying the following inequality:

$$e^{-\int_s^t a_i(u)du} \le K_i e^{-\int_s^t a_i^*(u)du}, \text{ for all } \; t, s, K_i \in \mathbb{R}, \; t - s > 0.$$

(H_5) Let

$$\max_{i,j \in I}\{K_i \frac{I_i^*}{\tilde{a}_i^*}\} := k, \; \max_{i \in I, A \in \Delta}\{\Big(\frac{1}{p(\tilde{a}_i^*)}\Big)^{\frac{1}{p}} 2^m \Omega_i^A\} = \rho < 1,$$
$$\max_{i \in I, A \in \Delta}\{\Big(\frac{1}{p(\tilde{a}_i^*)}\Big)^{\frac{1}{p}} 2^m \Omega_i'^A\} = \delta, \; \frac{\delta k}{1 - \delta} < 1,$$

where

$$\Omega_i^A = \max_{A \in \Delta}\Big((c_i^* \eta_i^*)^{\frac{1}{p}} + \sum_B a_{ij}^{A \cdot \bar{B}}(t) L_j^g + \sum_{l=1}^n \sum_B b_{ijl}^{A \cdot \bar{B}}(t)(L_j^g + L_l^g)\Big),$$
$$\Omega_i'^A = \Big(\frac{1}{p\tilde{a}_i^*}\Big)^{\frac{1}{p}} \Big(c_i^* \eta_i^* + 2^m \sum_{j=1}^n \big(\sum_B {a_{ij}^*}^{A \cdot \bar{B}} L_j^f \sum_{l=1}^n \sum_B {b_{ijl}^*}^{A \cdot \bar{B}} L_j^g L_j^l\big)\Big),$$

and $\varphi^* = \sup_{t \in \mathbb{R}} |\varphi(t)|, \; \varphi_* = \inf_{t \in \mathbb{R}} |\varphi(t)|$.

Using similar ideas as in Zhang [17], one can easily show the following result.

LEMMA 3.1. *Assume that that* $(H_1) \sim (H_5)$ *hold, we define the nonlinear operator* $(\Lambda_\varphi)_i^A(t)$ *by setting,*

$$(\Lambda_\varphi)_i^A(t) \;\; = \;\; \int_{-\infty}^t e^{-\int_s^t a_i(u)du} (\Gamma_\varphi)_i^A(s)ds$$

where

$$\begin{aligned}(\Gamma_\varphi)_i^A(t) &= \Big[c_i(s)\int_{t-\eta_i(s)}^{t}\varphi_i'^A(u)du + \sum_{j=1}^{n}\sum_{B} a_{ij}^{A\cdot\bar{B}}(s)\\ &\cdot \quad g_j^B(\varphi_j(s-\tau_{ij}(t))) + \sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{B} b_{ij}^{A\cdot\bar{B}}(s)\\ &\cdot \quad g_j^B(\varphi_j(s-\sigma_{ijl}(s)))g_l^B(\varphi_l(s-\mu_{ijl}(t))) + I_i^A(s)\Big]ds.\end{aligned}$$

Then $(\Lambda_\varphi)_i^A(t)$ maps into itself in the region $\mathcal{D}$.

Proof. Applying hypothesis $(H_1) \sim (H_5)$, the theorems and lemmas given previous, we can easily deduce that system (1) has a unique bounded solution (2). Let define a mapping Λ from $\mathcal{D}$ into itself by given $(\Lambda_\varphi)(t) = \{(\Gamma_\varphi)_i^A(t)\}$, for any $\varphi \in \mathcal{D}$. We show $(\Lambda_\varphi)_i^A(t)$ is a self mapping of from $\mathcal{D}$. Since $(\Gamma_\varphi)_i^A(t)$ Stepanov weighted pseudo almost periodic, we can decompose as $(\Gamma_\varphi)_i^A(t) = F_i(t) + G_i(t)$, where $F_i^b \in AP(\mathbb{R}, L^p([0,1],\mathcal{A})$ and $G_i^b \in PAP_0(\mathbb{R}, L^p([0,1],\mathcal{A})$. Furthermore, since $F_i \in S^pWPAP(\mathbb{R},\mathcal{A})$, for given $\epsilon_i > 0$, there exists $l_\epsilon > 0$ such that every interval of length l contains a number $\tau \in [t, t+l_{\epsilon_i}]$ such that

$$\sup_{t\in\mathbb{R}}\Big[\int_t^{t+1}||F_i(t+\tau) - F_i(t)||^p ds\Big]^{\frac{1}{p}} < \epsilon_i$$

By hypothesis and Lemmas, arguing as in the proof of ([3] Theorem 2), one can show easily that

$$\begin{aligned}&\sup_{t\in R}\Big[\int_t^{t+1}||\Psi_i(s+\tau) - \Psi_i(s)||^p ds\Big]^{\frac{1}{p}}\rho(t)dt\\ &\leq \epsilon_i.\end{aligned}$$

Hence we get that $\Psi_i \in S^pWPAP(\mathbb{R},\mathcal{A})$.

Next we show that $\Phi_i \in S^pWPAP_0(\mathbb{R},\mathcal{A})$:

$$\lim_{T\to\infty}\frac{1}{\mu(-T,T)}\int_{-T}^{T}\Big(\int_t^{t+1}||\Phi_i(s)||^p ds\Big)^{\frac{1}{p}}\rho(t)dt = 0.$$

By similar method in the proof of [15, Lemma 2.2] and definition, we have

$$\begin{aligned} & \frac{1}{\mu(T,\rho)}\int_{-T}^{T}\Big[\sup_{t\in R}\int_{t}^{t+1}|\int_{-\infty}^{t}e^{-\int_s^t a_i(u)du}\Phi_i(s)ds|^p du\Big]^{\frac{1}{p}}\rho(t)dt \\ = & \frac{1}{\mu(T,\rho)}\int_{-T}^{T}\Big[\sup_{t\in R}\int_{t}^{t+1}|\int_{0}^{\infty}e^{-\tilde{a}_i\sigma}\Phi_i(t-\sigma)d\sigma|^p du\Big]^{\frac{1}{p}}\rho(t)dt \\ \le & \frac{1}{\mu(T,\rho)}\int_{-T}^{T}[\sup_{t\in R}\int_{0}^{\infty}e^{-\tilde{a}_i\sigma}|\int_{t}^{t+1}|\Phi_i(t-\sigma)d\sigma|^p dt\Big]^{\frac{1}{p}}\rho(t)dt \\ \le & \Big(\frac{K_i}{p\tilde{a}_i^*}\Big)^{\frac{1}{p}}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\Big[\int_{t}^{t+1}|\Phi_i(t-\sigma)d\sigma|^p dt\Big]^{\frac{1}{p}}\rho(t)dt \\ \le & \epsilon_i\Big(\frac{K_i}{p\tilde{a}_i^*}\Big)^{\frac{1}{p}}. \end{aligned}$$

Thus, $\Phi_i(s)\in S^pWPAP_0$,
and so

$$\begin{aligned} & (\Lambda_\varphi)_i^A(t) \\ = & \int_{-\infty}^{t}e^{-\int_s^t a_i(u)du}F_i(s)ds+\int_{-\infty}^{t}e^{-\int_s^t a_i(u)du}G_i(s)ds \\ = & \Psi_i(s)+\Phi_i(s) \\ \in & \mathcal{D}, \end{aligned}$$

which implies that Λ_φ maps $\mathcal{D}$ into itself.

Furthermore, for all $\varphi\in\mathcal{D}$, set

$$\mathcal{D}^*=\{\varphi\mid\varphi\in\mathcal{D},\ ||\varphi-\varphi_0||_{S^p}\le\frac{\delta k}{1-\delta}\},$$

then, since $(\varphi_0)_i^A(t)=\int_{-\infty}^t e^{-\int_s^t a_i(u)du}I_i^A(s)ds$,
we can easily obtain

$$||\varphi_0||_{S^p}\le\max_{i,j\in I}\{K_i\frac{I_i^*}{\tilde{a}_i^*}\}:=k$$

and
$||\varphi||_{S^P}\le||\varphi-\varphi_0||_{S^p}+||\varphi_0||_{S^p}\le\frac{\delta k}{1-\delta}+k=\frac{k}{1-\delta}$, for all $\varphi\in\mathcal{D}^*$.

Using the Minkowski's inequality,
we obtain

$$
\begin{aligned}
& ||(\Lambda\varphi)_i^A - (\varphi_0)_i^A||_{S^p} \\
\leq\ & \sup_{t\in\mathbb{R}} \Big[\int_t^{t+1} || \int_{-\infty}^{\theta} e^{-\int_s^\theta a_i(u)du} \Big[c_i(s) \int_{s-\eta_i(s)}^{s} \varphi_i'^A(s)ds + \sum_{j=1}^{n}\sum_{B} a_{ij}^{A\cdot\bar{B}}(s) \\
\cdot\ & f_j^B(\varphi_j(s-\tau_{ij}(s))) + \sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{B} b_{ijl}^{A\cdot\bar{B}}(s) g_j^B(\varphi_j(s-\sigma_{ijl}(s))) g_l^B(\varphi_l(s) \\
& -\mu_{ijl}(s)))ds||^p d\theta\Big]^{\frac{1}{p}} \leq \sup_{t\in\mathbb{R}} \Big[\int_t^{t+1} || \int_{-\infty}^{\theta} e^{-\int_s^\theta a_i(u)du} c_i(s) \int_{s-\eta_i(s)}^{s} \varphi_i'^A(s) \\
& ds||^p d\theta\Big]^{\frac{1}{p}} + \sum_{j=1}^{n} \Big[\int_t^{t+1} || \int_{-\infty}^{\theta} e^{-\int_s^\theta a_i(u)du} \sum_{B} a_{ij}^{*\,A\cdot\bar{B}}(s) f_j^B(\varphi_j(s-\tau_{ij}(s)) \\
&)ds||^p d\theta\Big]^{\frac{1}{p}} + \sum_{j=1}^{n} \Big[\int_t^{t+1} || \int_{-\infty}^{\theta} e^{-\int_s^\theta a_i(u)du} \sum_{l=1}^{n}\sum_{B} b_{ijl}^{*\,A\cdot\bar{B}}(s) g_j^B(\varphi_j(s- \\
& \sigma_{ijl}(s))) g_l^B(\varphi_l - \mu_{ijl}(s)))ds||^{\frac{1}{p}} d\theta\Big]^{\frac{1}{p}} \leq \sup_{t\in\mathbb{R}} [\int_t^{t+1} \int_0^{\infty} e^{-p\tilde{a}_i\xi} c_i^* \eta^* ||\varphi||_{S^p} \\
+\ & \sum_{j=1}^{n}\sum_{B} a_{ij}^{*\,A\cdot\bar{B}} L_j^f \sum_{C} \int_t^{t+1} \int_0^{\infty} e^{-p\tilde{a}_i\xi} ||\varphi_i^C(\theta-\xi-\tau_{ij}(\theta-\xi))d\xi||^p d\theta]^{\frac{1}{p}} \\
+\ & \sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{B} b_{ijl}^{*\,A\cdot\bar{B}} L_j^g \sum_{C} \int_t^{t+1} || \int_0^{\infty} e^{-p\tilde{a}_i\xi} (\varphi_l^C(\theta-\xi-\sigma_{ij}(\theta-\xi))) \\
& L_l^g(\varphi_j^C(\theta-\xi-\mu_{ijl}(\theta-\xi)))d\xi|^p d\theta\Big]^{\frac{1}{p}} \leq \Big(\frac{1}{p\tilde{a}_i^*}\Big)^{\frac{1}{p}} c_i^* \eta_i^* ||\varphi||_{S^p} + 2^m \sum_{j=1}^{n} \\
& \big(\sum_{B} a_{ij}^{*\,A\cdot\bar{B}} L_j^f + \sum_{l=1}^{n}\sum_{B} b_{ijl}^{*\,A\cdot\bar{B}} L_j^g L_j^l\big) ||\varphi||_{S^p} \leq \Big(\frac{1}{p\tilde{a}_i^*}\Big)^{\frac{1}{p}} \Big(c_i^*\eta_i^* + 2^m \\
& \sum_{j=1}^{n} \big(\sum_{B} a_{ij}^{*\,A\cdot\bar{B}} L_j^f + \sum_{l=1}^{n}\sum_{B} b_{ijl}^{*\,A\cdot\bar{B}} L_j^g L_j^l)\big) ||\varphi||_{S^p} \\
\leq\ & \frac{\delta k}{1-\delta},
\end{aligned}
$$

which implies that $\Lambda\varphi \in \mathcal{D}$, consequently the mapping $\Lambda\varphi$ is a self mapping from $\mathcal{D}$ into itself. The proof is complete. $\square$

Now we give our main theorem. By arguing as in the verification and applying the similar mathematical analysis techniques of Theorem 3.1 in [14], we derive some new sufficient conditions ensuring the existence, uniqueness of weighted pseudo almost periodic solutions of system (2).

THEOREM 3.2. *Assume that* $(H_1) \sim (H_5)$ *hold, then system* (2) *has a unique* S^p*-almost periodic solution in the region* $\mathcal{D}^* = \{\varphi | \varphi \in \mathcal{D}, ||\varphi - \varphi_0||_{S^p} \leq \frac{\delta k}{1-\delta}\}$.

Proof. From the previous Lemmas and hypotheses, we see that the equation (2) has a unique weighted pseudo almost periodic solution as following

$$\begin{aligned}
(\Lambda_\varphi)_i^A(t) &= \int_{-\infty}^{t} e^{-\int_s^s a_i(u)du}\Big[c_i(s)\int_{s-\eta_i(s)}^{s}\varphi_i^{'}(s)du + \sum_{j=1}^{n}\sum_{B} a_{ij}^{A\cdot\bar{B}}(s) \\
&\cdot \quad g_j^B(\varphi_j(s-\tau_{ij}(s))) + \sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{B} b_{ijl}^{A\cdot\bar{B}}(s) \\
&\cdot \quad g_j^B(\varphi_j(s-\sigma_{ijl}(s)))g_l^B(\varphi_l(s-\mu_{ijl}(s))) + I_i^A(s)\Big]dt
\end{aligned}$$

Define a mapping $\Lambda : \mathcal{D}^* \to \mathcal{D}^*$ by given (Λ_φ).
By using the Minkowski's inequality and above definition, we have

$$\begin{aligned}
&||(\Lambda\varphi)_i^A - (\Lambda\psi)_i^A||_{S^p} \\
\leq\ & \sup_{t\in\mathbb{R}}\Big[\int_t^{t+1} ||\int_{-\infty}^{\theta} e^{-\int_s^\theta a_i(u)du}\Big[c_i(s)\int_{s-\eta_i(s)}^{s}(\varphi_i^{'A}(s) - \psi_i^{'A}(s))ds \\
&+ \sum_{j=1}^{n}\sum_{B} a_{ij}^{A\cdot\bar{B}}(s)\cdot f_j^B(\varphi_j(s-\tau_{ij}(s)) - \psi_j(s-\tau_{ij}(s))) + \sum_{j=1}^{n}\sum_{l=1}^{n} \\
&\sum_{B} b_{ijl}^{A\cdot\bar{B}}(s)g_j^B(\varphi_j(s-\sigma_{ijl}(s)) - \psi_j(s-\sigma_{ijl}(s)))g_l^B(\varphi_l(s-\mu_{ijl}(s)) \\
&-\psi_l(s-\mu_{ijl}(s)))\Big]ds||^p d\theta\Big]^{\frac{1}{p}} \leq \sup_{t\in\mathbb{R}}\Big[\int_t^{t+1} ||\int_{-\infty}^{\theta} e^{-\int_s^\theta a_i(u)du}c_i(s) \\
&\int_{s-\eta_i(s)}^{s}(\varphi_i^{'A}(s) - \psi_i^{'A}(s))ds||^p d\theta\Big]^{\frac{1}{p}} + \sum_{j=1}^{n}\Big[\int_t^{t+1} ||\int_{-\infty}^{\theta} e^{-\int_s^\theta a_i(u)du} \\
&\sum_{B} {a_{ij}^*}^{A\cdot\bar{B}}(s)f_j^B(\varphi_j(s-\sigma_{ij}(s)) - \psi_j(s-\sigma_{ij}(s))ds||^p d\theta\Big]^{\frac{1}{p}} +
\end{aligned}$$

$$\sum_{j=1}^{n}\Big[\int_t^{t+1}||\int_{-\infty}^{\theta}e^{-\int_s^{\theta}a_i(u)du}\sum_{l=1}^{n}\sum_{B}{b_{ijl}^{*}}^{A\cdot\bar{B}}(s)g_j^B(\varphi_j(s-\sigma_{ijl}(s))-$$

$$\psi_j(s-\sigma_{ijl}(s)))g_l^B(\varphi_l-\mu_{ijl}(s))-\psi_l(s-\sigma_{ijl}(s)))ds||^p d\theta\Big]^{\frac{1}{p}}\leq\sup_{t\in R}$$

$$[\int_t^{t+1}\int_0^{\infty}e^{-p\tilde{a}_i\xi}c_i^*\eta^*||\varphi-\psi||_{S^p}+\sum_{j=1}^{n}\sum_{B}{a_{ij}^{*}}^{A\cdot\bar{B}}L_j^f\sum_{C}\Big[\int_t^{t+1}\int_0^{\infty}$$

$$e^{-p\tilde{a}_i\xi}||\varphi_i^C(\theta-\xi-\tau_{ij}(\theta-\xi))-\psi_i^C(\theta-\xi-\tau_{ij}(\theta-\xi))d\xi||^p d\theta\Big]^{\frac{1}{p}}+$$

$$\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{B}{b_{ijl}^{*}}^{A\cdot\bar{B}}L_j^g\sum_{C}\Big[\int_t^{t+1}||\int_0^{\infty}e^{-p\tilde{a}_i\xi}L_i^g\Big(\varphi_i^C(\theta-\xi-\sigma_{ijl}(\theta-$$

$$\xi))-\psi_i^C(\theta-\xi-\sigma_{ijl}(\theta-\xi))\Big)L_l^g\Big(\varphi_l^C(\theta-\xi-\mu_{ijl}(\theta-\xi))-\psi_l^C(\theta-$$

$$\xi-\mu_{ijl}(\theta-\xi))\Big)d\xi||^p d\theta\Big]^{\frac{1}{p}}\leq\Big(\frac{1}{p\tilde{a}_i^*}\Big)^{\frac{1}{p}}\Big(c_i^*\eta_i^*||\varphi-\psi||_{S^p}+2^m\sum_{j=1}^{n}(\sum_{B}$$

$$a_{ij}^{*\,A\cdot\bar{B}}L_j^f+\sum_{l=1}^{n}\sum_{B}{b_{ijl}^{*}}^{A\cdot\bar{B}}L_j^gL_j^l)\Big)\cdot||\varphi-\psi||_{S^p}\leq\Big(\frac{1}{p\tilde{a}_i^*}\Big)^{\frac{1}{p}}(c_i^*\eta_i^*+2^m$$

$$\cdot\quad\sum_{j=1}^{n}(\sum_{B}{a_{ij}^{*}}^{A\cdot\bar{B}}L_j^f\sum_{l=1}^{n}\sum_{B}{b_{ijl}^{*}}^{A\cdot\bar{B}}L_j^gL_j^l)\cdot||\varphi-\psi||_{S^p}$$

$$\leq\quad\frac{\delta k}{1-\delta}||\varphi-\psi||_{S^p}\leq\rho||\varphi-\psi||_{S^p}$$

Since $\rho<1$, it implies that $\Lambda:\mathcal{D}^*\to\mathcal{D}^*$ is a contraction mapping. By contraction mapping principle of the $\mathcal{D}^*$, we obtain that the mapping Λ has a unique fixed point $z\in\mathcal{D}^*$ such that $\Lambda z=z$ which means that the equation (2) has a unique weighted pseudo almost periodic solution. The proof of the theorem is completed. □

4. Examples

In this section we consider a simple application of our abstracts results we give an modified example [1], [12] for $n=m=2$ as the following

Hopfield neural networks with time-varing leakage delays.

$$\begin{aligned} x'(t) &= -c_i(t)x_i(t-\eta_i(t)) + \sum_{j=1}^{2} a_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) \cdots\cdots (3) \\ &+ \sum_{j=1}^{2}\sum_{l=1}^{2} b_{ijl}(t)g_j(x_j(t-\sigma_{ijl}(t)))g_l(x_j(t-\mu_{ijl}(t))) + I_i(t) \end{aligned}$$

where

$$\begin{aligned} & a_1(t) = 0.1 + 0.2t|\cos\sqrt{3}t|,\ a_2(t) = 1.7 + 0.3|\sin\sqrt{2}t|, \\ & f_1(x) = (|x^0+1| - |x^2-1|)e_0 + e_1 \sin\frac{\sqrt{2}}{2}(x^1 + x^{12}) - e_2 \sin x^2 \\ & + e_{12}\tanh(x^2 + x^{12} + x^0)), e_0 = c, e_1 = i, e_2 = j, e_{12} = k. \end{aligned}$$

By a simple calculation, we can show easily that all the conditions in our main Theorem 3.2 are satisfied, which means the existence unique Spepanov weighted pseudo almost periodic solution of (3).

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